# Math 3280 Tutorial 3

Conditional Probability:

$$P(E|F) = P(EF)/P(F) \quad \text{if } P(F) > 0.$$

$$P(E) = P(EF^c) + P(EF) = P(E)\cdot P(E|F^c) + P(E)\cdot P(E|F).$$

Multiplicative rule:

$$P(E_1E_2\cdots E_n) = P(E_1)\cdot P(E_2|E_1)\cdot P(E_3|E_1E_2)\cdots\cdots P(E_n|E_1E_2\cdots E_{n-1}).$$

Independent events:

If $P(E|F) = P(E)$, $E$ is independent of $F$.

$\Leftrightarrow$ $P(EF) = P(E)\cdot P(E)$.

1. (example 2a).

A student is taking a one-hour-time-limit makeup examination. Suppose the probability that the student will finish the exam in less than $x$ hours is $x/2$, $x \in [0,1]$. Then given that the student is still working after 0.75 hour, what is the conditional probability that the full hour is used?

Solution: $E$: the full hour is used.

$F$: the student is still working after 0.75 hour.

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

$EF$: the full hour is used.

$P(EF) = 1 - P(\text{student finish exam in 1 hour})$

$= 1 - \frac{1}{2} = 0.5$

$P(F) = 1 - P(\text{student finish in 0.75 hour})$

$= 1 - \frac{0.75}{2} = 0.625$

$P(E|F) = 0.5/0.625 = 0.8.$

2. (4f). An infinite sequence of independent trials are to be performed. Each trial results in a success with probability $p$ and failure with probability $1-p$.

(a) at least 1 success in the first $n$ trials.

(b) exactly $k$ successes occur in first $n$ trials.

Solution: $E_i$ denote the event of failure in the $i$-th trial.

(a). $P$(at least 1 success in first $n$ trials) $\qquad P(E_i) = 1-P.$

$= 1 - P$(no success in first $n$ trials)

$= 1 - P(E_1 E_2 \cdots E_n)$

$= 1 - P(E_1) \cdot P(E_2|E_1) \cdot P(E_3|E_1E_2) \cdots P(E_n|E_1 \cdots E_{n-1})$

$= 1 - P(E_1) \cdot P(E_2) \cdot P(E_3) \cdots P(E_n)$

$= 1 - (1-P)^n$

(b). $\binom{n}{k}^{\text{choose } k} \cdot (P)^k \cdot (1-P)^{n-k}$ (remaining $n-k$)

3. (example 4i). There are $n$ coupons, and each new one collected is independently of type $i$ with probability $P_i$, $\sum_{i=1}^{n} P_i = 1$. Suppose $k$ coupons are to be collected. $A_i$ is the event that at least one type $i$ coupon among these collected. then for $i \neq j$,

find $P(A_i)$, $P(A_i \cup A_j)$, $P(A_i | A_j)$.

Solution: $P(A_i) = 1 - P(A_i^c) = 1 - P$(no type $i$ coupon)

$= 1 - (1-P_i)^k$

$P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i A_j)$

$P(A_i \cup A_j) = P(\text{at least one type } i \text{ or type } j \text{ coupon})$

$= 1 - P((A_i \cup A_j)^c)$

$= 1 - P(\text{no type } i \text{ or type } j \text{ coupon})$

$= 1 - (1-P_i-P_j)^k$

$P(A_i | A_j) = \underline{P(A_i A_j)} / \underline{P(A_j)}$

$P(A_i A_j) = P(A_i) + P(A_j) - P(A_i \cup A_j)$

$= 1-(1-P_i)^k + 1-(1-P_j)^k - (1-(1-P_i-P_j)^k)$

$P(A_i | A_j) = \ldots$

4. Independent trials that result in a success with probability $p$ and a failure with probability $(1-p)$ are called Bernoulli trials. Let $P_n$ denote the probability that $n$ Bernoulli trials result in an even number of success ( $0$ is an even number) Show that

$$P_n = p(1-P_{n-1}) + (1-p) \cdot P_{n-1}, \quad n \geq 1.$$

and

$$\boxed{P_n = \frac{1+(1-2p)^n}{2}}$$

$E_n$ & $E_{n-1}$.

Solution: $E_n$: n trials result in an even number of success

$P(E_n) = P_n$.

$P(E_n) = P(E_n E_{n-1}) + P(E_n E_{n-1}^c)$

$= \underbrace{P(E_{n-1})}_{P_{n-1}} P(E_n | E_{n-1}) + \underbrace{P(E_{n-1}^c)}_{(1-P_{n-1})} P(E_n | E_{n-1}^c)$

$P(E_n | E_{n-1}) = P(\text{the result of } n\text{-th trial is failure})$

$= 1-p$

$P(E_n | E_{n-1}^c) = P(\text{the result of } n\text{-th trial is success})$

$= p.$

$E_n$: n trials even number of success

$E_{n-1}$: n-1 trials odd number of success.

$$P_n = P_{n-1}(1-p) + (1-P_{n-1})\cdot p.$$

$$P_n = (1-2p)\cdot P_{n-1} + p.$$

$(P_n - x) = y_n$ is a geometric series.

parameter.

$y_n = y\cdot y_{n-1}$. $y$ is a parameter

$$(P_n - x) = y\cdot(P_{n-1} - x). \Rightarrow y = 1-2p.$$

$$(1-y)\cdot x = p \Rightarrow x = \frac{1}{2}.$$

$$P_n - x = y\cdot(P_{n-1} - x).$$

$$P_n = (1-2p)\cdot P_{n-1} + p$$

$$P_n = y\cdot P_{n-1} + x - xy$$

$y = 1-2p.$

$p = x - xy.$

$$\left(P_n - \frac{1}{2}\right) = (1-2p)\cdot\left(P_{n-1} - \frac{1}{2}\right). \Rightarrow P_n - \frac{1}{2} = (1-2p)^{n-1}\cdot\left(P_1 - \frac{1}{2}\right).$$

$P_1 = P(1 \text{ trial result in an even number of success})$

$= 1-p$

$$P_n - \frac{1}{2} = (1-2p)^{n-1}\cdot\left(1-p-\frac{1}{2}\right)$$

$$= \frac{1}{2}\cdot(1-2p)^n.$$

$$\Rightarrow P_n = \frac{1+(1-2p)^n}{2}$$